

Finite volume corrections and low momentum cuts in the thermodynamics of quantum gases

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The conjecture, that the finite volume corrections to the thermodynamic functions can be correctly reproduced by using the thermodynamic limit with low particle momenta cutoff is examined in a very transparent example of an ideal boson gas in one dimension. We show that this conjecture is always true in principle, and derive convenient relations for the momentum cutoff dependence on thermal parameters in the asymptotic limits of large and small volume.

I. INTRODUCTION

In the phenomenological analysis of particle production in high energy heavy ion collisions it was shown that the thermal statistical models provide a very satisfactory description of particle yields measured in the experiments [1–5]. The statistical operator in these models is usually constructed as that for a mixture of ideal gases constrained by the conservation laws. The success of such statistical models has recently got a theoretical support from lattice QCD (LQCD), showing that they are also capable to quantify the equation of state and different fluctuation observables in the hadronic phase obtained in LQCD [6–9].

In heavy ion collisions, however, we are dealing with a finite system, thus in the thermal analysis of data, the effects related with the finite volume should be accounted for. Indeed, it was shown, that such corrections are crucial in small systems, and they have been quantified in the context of exact conservation laws [4, 5, 10]. Recently, the influence of the finite volume effects were also addressed in the context of fluctuation observables in the momentum space [11, 12]. There, it was conjectured, that the final volume corrections in the thermodynamic observables can be correctly reproduced by using the thermodynamic limit with low particle momenta cutoff.

In the statistical thermodynamics of gases, in the thermodynamic limit, where the volume and the number of particles tend to infinity at fixed particle density and temperature, the thermodynamic functions are represented as integral over momentum space. Following numerical results of Refs. [11], [12] and [13], one expects that the finite volume corrections for the thermodynamics of quantum gases can be approximated by cutting off the low momentum regions in these integrals formulae.

In the present paper we use a very transparent model of one-dimensional gas of noninteracting bosons, to discuss this conjecture. We find that for a significant range of volumes the approximation is valid with a simple intuitively plausible momentum cutoff. For smaller volumes,

the finite volume effects can still be reproduced by introducing a momentum cutoff, but the position of the cutoff becomes a complicated function of the parameters of the system. For sufficiently large volume, the momentum cutoff $k_c = \pi/2L$ is found to be simply connected with the system size L , which makes it convenient for the phenomenological application of statistical models to the finite systems.

II. THERMODYNAMIC FUNCTIONS AT FINITE VOLUME

Let us consider an ideal gas of identical bosons of mass m enclosed in a one-dimensional box of length L . Assuming the boundary conditions for the single-particle wave function,

$$\psi(0) = \psi(L) = 0, \quad (1)$$

one finds, that each single-particle state is unambiguously defined by its energy,

$$E_n = \sqrt{\left(\frac{\pi n}{L}\right)^2 + m^2}; \quad n = 1, 2, \dots \quad (2)$$

All the thermodynamic functions of the gas can be obtained from the thermodynamic potential

$$\Omega(T, V, \mu) = T \sum_{n=1}^{\infty} \log \left(1 - e^{-\beta(E_n - \mu)} \right), \quad (3)$$

where, as usual, $\beta = T^{-1}$. It is convenient to introduce the function

$$\Omega_n(T, V, N) = T \log \left(1 - e^{-\beta(E_n - \mu)} \right), \quad (4)$$

where n is a continuous variable ranging from zero to infinity.

For L tending to infinity, the function $\Omega(T, V, \mu)$ goes over into the integral

$$\Omega_{int}(T, V, \mu) = T \int_0^\infty \Omega_n(T, V, \mu) dn. \quad (5)$$

Making the substitution, $z = \frac{\beta\pi}{L}n$, the potential can be also expressed, as

$$\Omega_{int}(T, V, \mu) = \frac{L}{\beta\pi} \int_0^\infty \log \left(1 - e^{-(E(z) - \beta\mu)} \right) dz, \quad (6)$$

where, $E(z) = \sqrt{z^2 + \beta^2 m^2}$.

Since Ω_{int} is proportional to the volume L , the thermodynamic identity

$$p = - \left(\frac{\partial \Omega}{\partial L} \right) \quad (7)$$

yields, in the integral approximation, a well-known relation,

$$p_{int} = - \frac{1}{L} \Omega_{int}, \quad (8)$$

for the thermodynamic pressure in one-dimension.

A. The approximations of the final volume potential

In the following, we focus on the finite volume corrections to thermodynamics of an ideal boson gas and discuss their relation to the particle momentum cutoff. We examine different approximations of the potential (3) and consider them to be *good* if, for a given set of parameters, they differ by less than one percent from the exact result. For numerical calculations we take the particle mass $m = 140$ MeV, the temperature $T = 120$ MeV, and assume vanishing chemical potential $\mu = 0$.

In Fig. 1 we show the L -dependence of $\Omega(120, L, 0)$ and $\Omega_{int}(120, L, 0)$ from Eqs. (3) and (5), respectively. The approximation of the sum (3) by the integral (5) is valid at very large L , and it is *good* only at $L > 166$ fm. However, one finds that an approximation to Ω ,

$$\Omega_{cor} = \Omega_{int} - \frac{1}{2} \Omega_0. \quad (9)$$

is much better than Ω_{int} . This may be interpreted as an application of the Euler-Maclaurin series for the difference between the sum and the corresponding integral. Only one term in this series is different from zero. Thus, the series is convergent, but unfortunately not to the expected limit. The same formula can be obtained by applying the well-known method of trapezes to approximate the integral.

The correction term in Eq. (9), is calculated from Eq. (3), and at $T = 120$ MeV, one gets $\Omega_0 = -44.8$ MeV. As shown in Fig. 1, the improvement of the approximation of Ω by Ω_{cor} is very significant, and is much better than Ω_{int} , since it is *good* already for $L > 3.7$ fm.

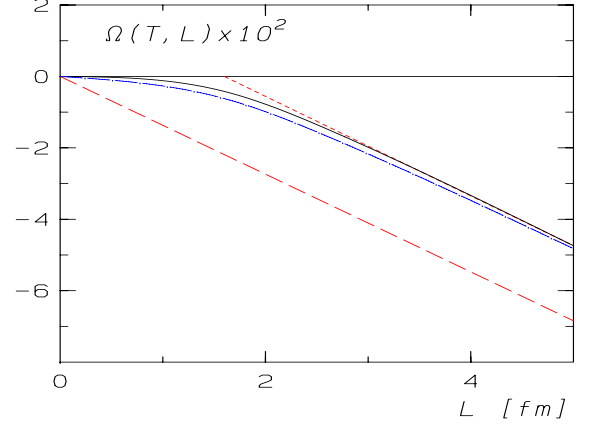


FIG. 1: Comparison of the thermodynamic potential $\Omega(120, L, 0)$ (full line) with its approximations: $\Omega_{int}(120, L, 0)$ (long-dashed line), $\Omega_{cor}(120, L, 0)$ (short-dashed line) and $\Omega_{cut}(120, L, 0, 0.5)$ (dashed-dotted line), *see text*.

1. Momentum cutoff and the finite volume effects

The thermodynamic potential of an ideal boson gas enclosed in a finite volume is a transparent example of a system where the conjecture, that by cutting off the low momenta in the thermodynamic limit one reproduces the exact results for the bounded system, can be verified. In general, we consider the problem of finding a cutoff n_c such that

$$\Omega_{cut}(T, V, \mu, n_c) \equiv T \int_{n_c}^\infty \Omega_n(T, V, \mu) dn = \Omega(T, V, \mu). \quad (10)$$

The function Ω_n is a continuous, monotonically increasing function of $\frac{n}{L}$. Therefore,

$$\int_n^{n-1} \Omega_m(T, L, \mu) dm < \Omega_n(T, L, \mu) < \int_n^{n+1} \Omega_m(T, L, \mu) dm, \quad (11)$$

and consequently

$$\Omega_{cut}(T, L, \mu, 0) < \Omega(T, L, \mu) < \Omega_{cut}(T, L, \mu, 1). \quad (12)$$

This implies that, at any given T , μ and L , it is always possible to find, in the range

$$0 < n_c < 1, \quad (13)$$

a value of n_c satisfying the relation (10).

We consider first a large- L limit, in which the difference between Ω_{int} and Ω_{cut} can be approximated by a linear function of n_c . Assuming that in this region Ω_{cor} is a *good*

approximation to Ω , one gets the intuitively expected result (see e.g. [11])

$$n_c = \frac{1}{2}. \quad (14)$$

The particle momentum $k_n = \frac{\pi}{L}n$, thus the cutoff at n_c is equivalent to the cutoff in the particle momentum at the corresponding

$$k_c = \frac{\pi}{2L}, \quad (15)$$

which is consistent with the numerical value found in Ref. [12] for the transverse momentum cutoff in three dimensions.

A comparison of Ω_{cut} for this cutoff with the exact result, and with the two other approximations, discussed in the previous section, is shown in Fig. 1. The approximation of $\Omega(120, L, 0)$ by $\Omega_{cut}(120, L, 0, \frac{1}{2})$ is *good* for $L > 5.6$ fm. Actually, the approximation to the correction term Ω_0 is *good* only for $L > 8.4$ fm, but with increasing L the relative importance of the correction term decreases.

Another transparent expression for n_c can be obtained in the $L \rightarrow 0$ limit, where the ratio $\frac{L}{n_c}$ is so small that the approximation,

$$\log(1 - e^{-\beta(E_n - \mu)}) \approx -e^{\beta(\frac{n\pi}{L} - \mu)} \quad (16)$$

is valid. For the values of the parameters used in the previous section, this is the case when $L < 0.076n_c$ fm. Then, the condition (10) for n_c can be approximated by

$$\int_{n_c}^{\infty} e^{-\frac{\beta\pi n}{L}} dn = e^{-\frac{\beta\pi}{L}} \quad (17)$$

and yields

$$n_c = 1 + \frac{L}{\beta\pi} \log \frac{L}{\beta\pi}. \quad (18)$$

The requirement of consistency with the condition (16) gives the limitation $L < 0.076$ fm. Thus, this low- L formula (18) is of little interest for direct physical applications.

The numerical results for n_c as a function of L are shown in Fig. 2. It is seen, that n_c decreases monotonically from unity at $L \rightarrow 0$ to its asymptotic value $n_c = \frac{1}{2}$ at large L . For higher temperatures, the validity limit of this approximation is shifted to lower values of L .

The results for the number of particles, energy and entropy are qualitatively similar to the results for the potential Ω . They can be obtained either by studying the exact expressions for these quantities, or by using the results for the potential Ω and suitable thermodynamic identities.

Note that some identities derived using specific assumptions about the system, e.g. $\Omega = -pL$, cannot be used blindly. For the pressure, the situation is somewhat different. Using Eq. (7) one gets

$$p(T, L, \mu) = \frac{\pi^2}{L^3} \sum_{n=1}^{\infty} \frac{n^2}{(e^{\beta(E_n - \mu)} - 1) E_n}. \quad (19)$$

Defining the function of n ,

$$p_n(T, L, \mu) = \frac{\pi^2}{L^3} \frac{n^2}{(e^{\beta(E_n - \mu)} - 1) E_n} \quad (20)$$

we see that $p_0(T, L, \mu) = 0$, thus no subtraction should be made. An alternative derivation of this result follows from the remark, valid in the region where Ω_{cor} is a *good* approximation to Ω , that the difference between Ω_{int} and Ω does not depend on L , and thus its derivative with respect to L does not contribute to the pressure. Consequently, the integral approximation

$$p_{int}(T, L, \mu) = \int_0^{\infty} p_n(T, L, \mu) dn = \frac{1}{\pi} \int_0^{\infty} \frac{z^2 dz}{(e^{(E(z) - \beta\mu)} - 1) E(z)} \quad (21)$$

is much better than in the previous case of the thermodynamic potential.

The L dependence of p_{int} and p is compared in Fig. 2. The integral approximation is *good* in the whole region where $L > 4.4$ fm. Note, that since Ω_{cor} is a good approximation to Ω in this region, we get from Eq. (8) the relation

$$\Omega + \frac{1}{2}\Omega_0 \approx -pL. \quad (22)$$

The numerical results for the cutoff parameter and the limits of the applicability of different approximations of the finite volume thermodynamic functions were calculated at fixed temperature. However, the temperature dependence of the momentum cutoff function is straightforward.

III. CONCLUSIONS

For an ideal boson gas in one-dimension, we have demonstrated that the finite volume corrections to thermodynamic functions can be exactly reproduced in the thermodynamical limit by making suitable low-momentum cuts in the particle phase-space.

The calculations with the cutoff parameter are greatly simplified when it does not depend on the model parameters. We have shown, that for the one-dimensional model considered here, this is the case when the length of the

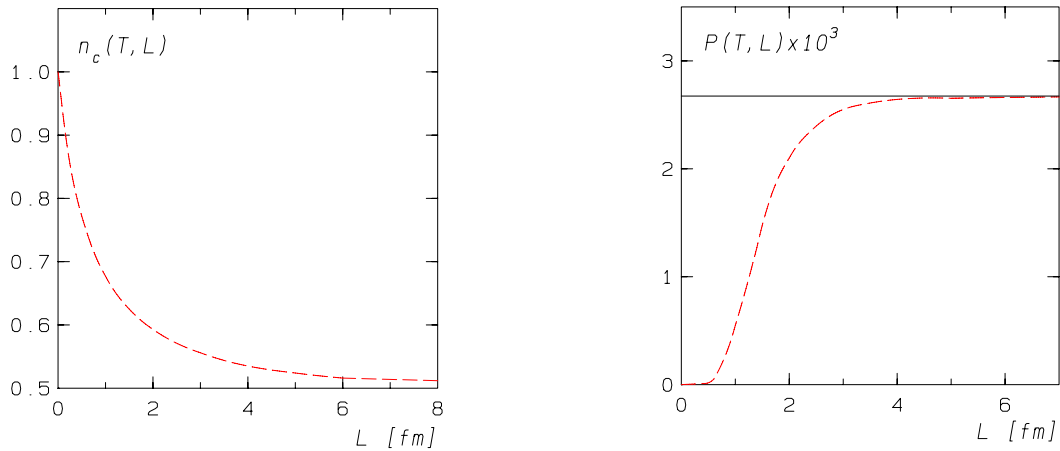


FIG. 2: Left-hand figure: The L -dependence of the cutoff parameter n_c . Right-hand figure: Comparison of the function $p(120, L, 0)$ (dashed line) with the approximation $p_{int}(120, L, 0)$ (full line).

vessel L is no less than a few fermis. There, the momentum cutoff $k_c = \pi/2L$ is simply related with the volume, which makes it very transparent in the numerical analysis of finite volume effects in different observables. For smaller volumes, the finite volume effects can still be reproduced by introducing a momentum cutoff, but the position of the cutoff becomes a more complicated function of the parameters of the system. Nevertheless, in the limit of $L \rightarrow 0$, a compact analytic expression of the momentum cutoff has been derived.

The above results support the conjecture that in the application of statistical models to particle production in heavy ion collisions the importance of the finite volume effects can be studied in the thermodynamic limit by implementing the cutoff in the momentum phase space of

particles. From the discussion presented here, it is transparent, that the finite volume corrections are relevant for small systems when their size is less than a few fermis. This is the case in a medium created in elementary, or peripheral nucleus-nucleus collisions.

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- [1] A. Andronic, Int. J. Mod. Phys. A **29**, 1430047 (2014).
 - [2] M. Floris, Nucl. Phys. A **931**, 103 (2014).
 - [3] A. Andronic, P. Braun-Munzinger and J. Stachel, Phys. Lett. B **673**, 142 (2009).
 - [4] P. Braun-Munzinger, K. Redlich, and J. Stachel, Particle production in heavy ion collisions, in: R. C. Hwa and X.-N. Wang (Eds.), Quark-Gluon Plasma 3, World Scientific, Singapore, 2004, pp. 491-599. e-Print: nucl-th/0304013.
 - [5] F. Becattini, P. Castorina, A. Milov and H. Satz, Eur. Phys. J. C **66**, 377 (2010).
 - [6] F. Karsch, J. Phys. G **38**, 124098 (2011).
 - [7] A. Bazavov *et al.* [HotQCD Collaboration], Phys. Rev. D **86**, 034509 (2012). A. Bazavov *et al.* [HotQCD Collaboration], Phys. Rev. D **90**, 094503 (2014).
 - [8] F. Karsch, Acta Phys. Polon. Supp. **7**, no. 1, 117 (2014).
 - [9] S. Borsanyi, Z. Fodor, S. D. Katz, S. Krieg, C. Ratti and K. K. Szabo, Phys. Rev. Lett. **111**, 062005 (2013).
 - [10] S. Hamieh, K. Redlich and A. Tounsi, Phys. Lett. B **486**, 61 (2000).
 - [11] A. Bhattacharyya, R. Ray, S. Samanta and S. Sur, Phys. Rev. C **91**, no. 4, 041901 (2015).
 - [12] F. Karsch, K. Morita and K. Redlich, Phys. Rev. C **93**, no. 3, 034907 (2016).
 - [13] J. Engels, F. Karsch and H. Satz, Nucl. Phys. B **205**, 239 (1982).